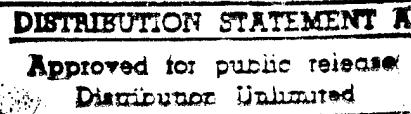




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Carrying a Ladder Under a Velocity Constraint

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Abstract We consider the problem of computing a time-optimal motion for two robots carrying a ladder from an initial position to a final position in a plane without obstacles. At any moment during the motion, the distance between the robots remains unchanged and the speed of each robot must be either a given constant v , or 0. A trivial lower bound on time for the robots to complete the motion is the time needed for the robot farther away from its destination to move to the destination along a straight line at a constant speed of v . This lower bound may or may not be achievable, however, since the other robot may not have sufficient time to complete the necessary rotation around the first robot (that is moving along a straight line at speed v) within the given time. We first derive, by solving an ordinary differential equation, a necessary and sufficient condition under which this lower bound is achievable. If the condition is satisfied, then a time-optimal motion of the robots is computed by solving another differential equation numerically. Next, we consider the case when this condition is not satisfied, and show that a time-optimal motion can be computed by taking the length of the trajectory of one of the robots as a functional and then applying

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the method of variational calculus. Several optimal paths that have been computed using the above methods are presented.

Keywords Motion planning, time-optimal motion, mobile robots

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1 Introduction

Suppose that two mobile robots are carrying a ladder, one at each end, in the plane without obstacles. What is the time-optimal motion of the robots from a given initial position to a final position of the ladder, subject to the given constraints on the kinematics of the robots, such as maximum acceleration and velocity? Note that the ladder enforces the distance between the robots to remain unchanged during the motion.

In this paper, we present a solution to the above problem for the case when the kinematic constraint of the robots is simply the following: At any moment during the motion, the speed of each robot is either a given constant v , or 0. That is, each robot must move at speed v whenever it decides to move, and it is not allowed to move at a nonzero speed less than v . The work reported here on this restricted case is considered a step towards the goal of solving the general optimization problem stated in the previous paragraph under more complex kinematic constraints (and possibly in the presence of obstacles).

The difficulty in finding an optimal motion even for the simple case considered here lies, intuitively, in “balancing” the trajectories of the two robots to achieve minimum time. For example, suppose that robots A and B must move to A' and B' , respectively, as is shown in Figure 1, carrying a ladder. (We use “ A ” and “ B ” to refer to the robots as well as their initial positions.) One way to move is to first rotate the ladder about its center (the midpoint of AB) until it is parallel to $A'B'$, and then translate it to $A'B'$, as is shown in Figure 2. This is a legal motion, since the distance between the robots remains unchanged and the robots can move at speed v all the time. Another strategy would be to do rotation and translation simultaneously, obtaining a motion such as the one shown in Figure 3. Note that the path of robot B is a curve, rather than a straight line segment, indicating that B “yields” to give just enough time for A to complete the necessary rotation. In fact, the motion shown in Figure 3 is a time-optimal motion obtained by the method given in Section 3.

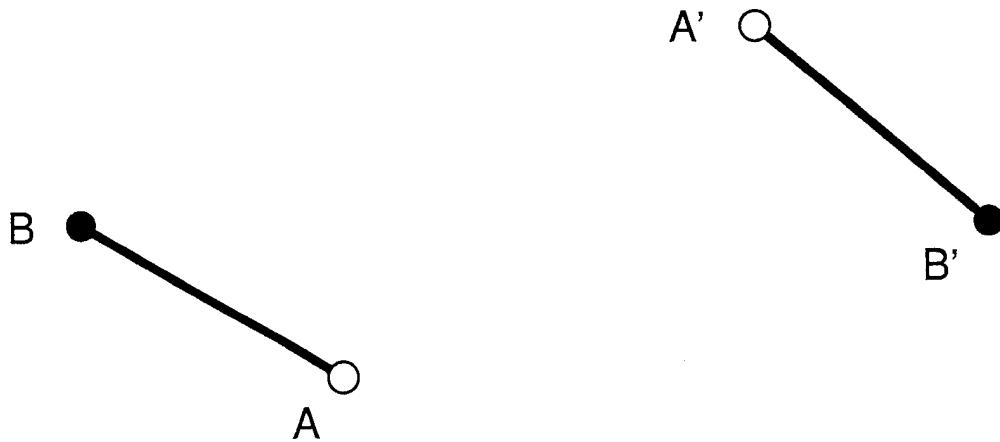


Figure 1: Robots A (empty circle) and B (filled circle) move to A' and B' , respectively, carrying a ladder.

The time optimization problem considered in this paper is related to the Ulam problem of optimal motion of a line segment [16]. Proposed by S. M. Ulam in 1960, the Ulam problem

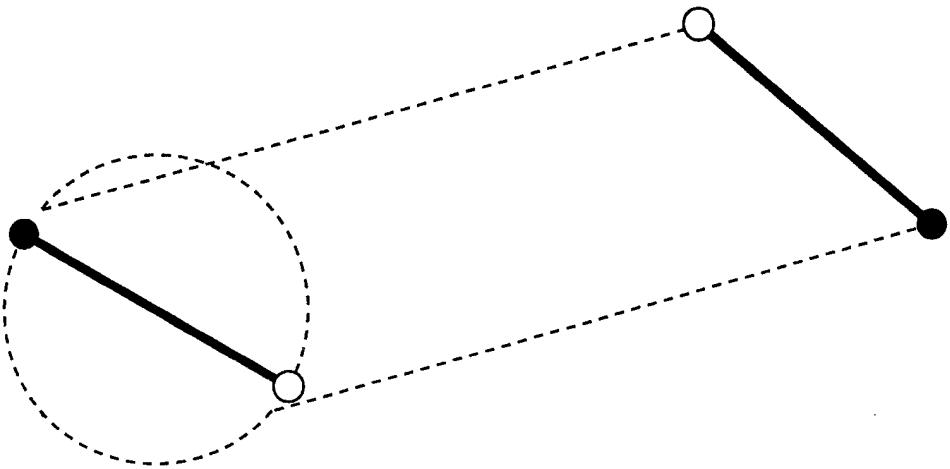


Figure 2: A suboptimal motion consisting of a rotation followed by a translation.

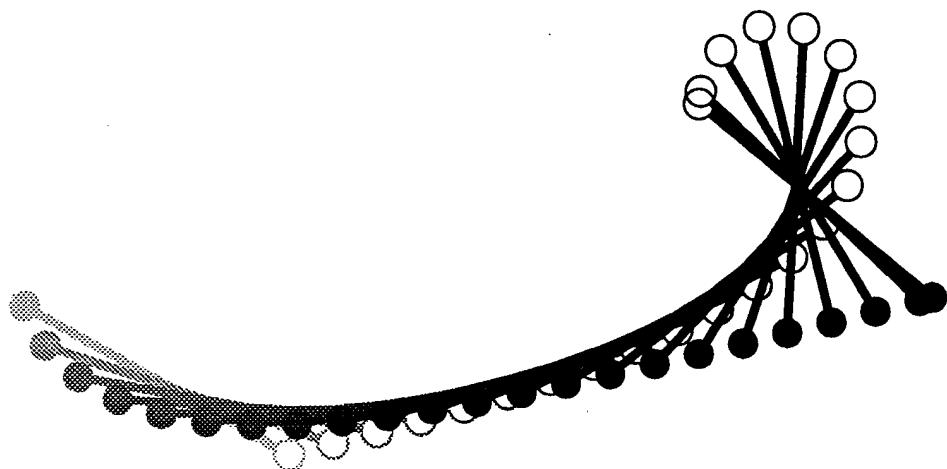


Figure 3: Time-optimal motion for the instance shown in Figure 1. The figure consists of a series of snapshots. In the first few shots the segment increases in grayscale as time increases.

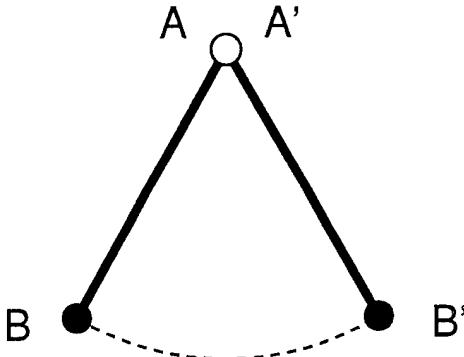


Figure 4: An instance in which the initial position of robot A is also its final position A' , where $\angle BAB' = 60^\circ$. The dashed line shows the arc that B will follow if A remains stationary.

asks for the general solution for moving a line segment to a given position while keeping the sum of the lengths of the trajectories of the two endpoints minimum. Gurevich [8] gives a solution to the Ulam problem using variational calculus and Euler-Lagrange equations [5]. A book by Dubovitskij [3] studies the Ulam problem with the two endpoints moving on prescribed surfaces in an n -dimensional space.

In general, the solution to the Ulam problem that minimizes the sum of the trajectories differs from the result of time optimization. Consider the instance shown in Figure 4 in which the initial position of robot A is also its final position. Our result given in Section 2 shows that the time-optimal motion for this instance is the one shown in Figure 5, in which robot B moves straight to B' at speed v while A makes a circular motion, with total time $|BB'|/v$. This motion, for which the sum of the lengths of the trajectories of the two robots is $2|BB'|$, is not optimal for the Ulam problem, since a simple rotation of the ladder about A (without moving robot A), as indicated in dashed line in Figure 4, gives a sum that equals the length of the arc from B to B' , which is smaller than $2|BB'|$.

There are other problems related to ours. Feinberg and Papadimitriou [4] consider the problem of finding possible paths for moving two points that must always remain a fixed distance apart, from initial to final positions among polygonal obstacles. Goodman, Pach and Yap [6] study the problem of deciding whether two painters carrying a ladder can stay on a specific path. These works consider the ladder moving problem in the plane in the presence of obstacles, with an emphasis on proving the feasibility of the constraint motion. In contrast, in our problem the robots are free to move in the entire plane, where feasibility is not a concern. Our goal is to select a time-optimal path from an infinite number of possible paths under the given velocity constraint. Naturally, the study of time-optimal motion in the presence of obstacles will not be possible without the understanding of our simpler problem. Another related work is by Mitchell and Wynters [10], who consider the motion of two covisible points (that need *not* remain a fixed distance apart) in the plane containing obstacles, that minimizes the sum of the lengths of the trajectories of the points, or the length of the longer trajectory. They also consider the problem of minimizing the

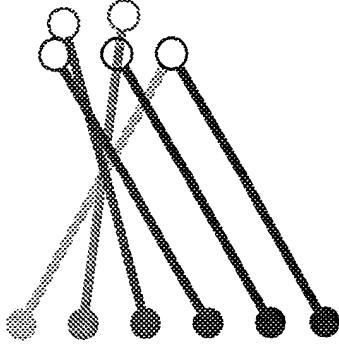


Figure 5: Time-optimal motion for the instance shown in Figure 4.

time of the motion of covisible points assuming a common upper bound on their velocity. Their assumption that the two points need only remain covisible gives a useful combinatorial structure that one can exploit in finding a desired optimal motion. Our assumption that the robots must remain a fixed distance apart, on the other hand, does not seem to yield such a property.

Other problems related to ours in a broader sense include formation and maintenance of geometric patterns by mobile robots in the plane, studied in the area of multiple distributed robot systems, where it is typically assumed that the robots are autonomous agents that make decisions individually using only local information available to them. We refer the reader to Sugihara and Suzuki [13], Suzuki and Yamashita [14, 15], and Chen and Luh [1] for discussions on formation and related agreement problems, Kawauchi, Inaba and Fukuda [9] for discussions on mobile cells carrying an object coordinatedly, and Parker [11] for the issue of balancing local and global control on maintaining a formation of a group of four robots.

A trivial lower bound on the time needed to move the ladder in an instance such as the one given in Figure 1 is $|BB'|/v$, assuming $|BB'| \geq |AA'|$, which is achievable when A completes the necessary rotation while B moves straight to B' at speed v . This lower bound may or may not be achievable, however, since A may not have sufficient time to complete the necessary rotation around B within time $|BB'|/v$. So first in Section 2, we derive, by solving an ordinary differential equation, a necessary and sufficient condition for this lower bound to be achievable. The condition is expressed in terms of the length $|BB'|$ and the angles that AA' and BB' make with the line passing through B and B' . If the condition is satisfied, then the desired optimal motion can be computed by solving another differential equation numerically. Next in Section 3, we derive a method for computing a time-optimal path when the condition is not satisfied, by taking the length of the trajectory of one of the robots as a functional and then applying variational calculus and Euler-Lagrange equations [5], techniques used by Gurevich in [8] to solve the Ulam problem. When applying the method, we first solve a set of equations involving the Legendre elliptic integrals [7], and then use the solutions obtained to numerically solve a set of differential equations that describe the motion of the robots. Time-optimal motions obtained using the results outlined above for

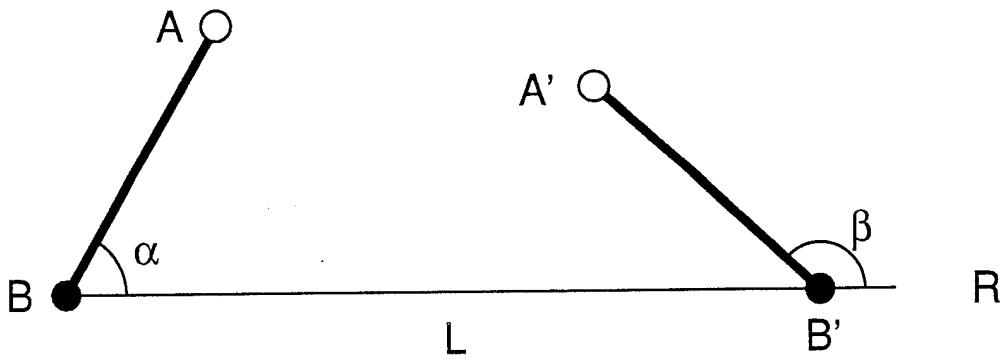


Figure 6: L , α and β .

several instances of the problem are also presented. Conclusions are found in Section 4.

2 Achieving a Lower Bound

2.1 A Lower Bound

Without loss of generality assume that $|BB'| \geq |AA'|$, as is shown in Figure 6. For convenience, we take AB to be of unit length ($|AB| = 1$), and let $L = |BB'|$ be the distance between B and B' . Since the robots can move only at speed v or 0 at any moment, a lower bound on the time it takes to move AB to $A'B'$ is

$$L/v.$$

Intuitively, this lower bound is achievable if A can complete the necessary rotation around B within time L/v while B moves straight to B' at speed v . The main result of this section is a necessary and sufficient condition for this to be possible.

Let R be the line passing through B and B' . Let α be the angle between AB and R , and β the angle between $A'B'$ and R , as is shown in Figure 6. Without loss of generality, we assume $0 \leq \alpha \leq 180^\circ$ and $-180^\circ < \beta \leq 180^\circ$.

The following lemma states that A cannot cross R if the bound L/v is to be achieved.

Lemma 1 *Suppose that B moves straight to B' at speed v and reaches B' at time L/v . Then during the time interval $[0, L/v]$, either A always lies on R , or A never lies on R .*

Proof We only give an outline and leave a formal proof to the reader. We can view the velocity of robot A as composed of two parts. The first part comes from translational motion, and is equal to the velocity of B with magnitude v . The second part is the rotational contribution. When A lies on R , the rotational part is perpendicular to the translational part, and unless the former is zero the value of the net velocity will exceed v . Therefore once A lies on R , it is not possible for A to rotate around B and leave R . For a similar reason, it is not possible for A to move to a point on R unless it is already on R . \square

Lemma 1 implies that AB can reach $A'B'$ within time L/v only if $\alpha = \beta = 0^\circ$, $\alpha = \beta = 180^\circ$, or A and A' are on the same side of R (i.e., $0 < \alpha, \beta < 180^\circ$). Since AB can simply be translated to $A'B'$ in time L/v whenever $\alpha = \beta$, in the following we only need to examine the case $0 < \alpha, \beta < 180^\circ$ and $\alpha \neq \beta$. In this case, the assumption $|BB'| \geq |AA'|$ implies $0 < \alpha < \beta < 180^\circ$.

It is convenient to view the motion in the coordinate system that moves with B at speed v . In this coordinate system, A can only rotate around B because $|AB| (= 1)$ is a constant. Let us denote by θ the angle between segment AB and R that changes with time, where $\alpha \leq \theta \leq \beta$. Of course, $\theta = \alpha$ at time 0, and $\theta = \beta$ at the end of a successful motion.

If we denote by ω the angular speed of A , then the linear speed of A due to rotation has the value ω since $|AB| = 1$. Since A also translates with B at speed v , the actual velocity of A in the laboratory coordinate system is the vector sum of the velocity due to rotation around B and the velocity due to translation with B . Writing the magnitude of this net velocity as V , we have

$$V^2 = v^2 + \omega^2 - 2v\omega \sin \theta. \quad (1)$$

Equation 1 is obtained by using the fact that the angle between the two component velocities is $\theta + 90^\circ$. Since both robots move at speed v , we have

$$V = v. \quad (2)$$

From Eq. 1 and Eq. 2 we obtain

$$\omega = 2v \sin \theta \quad (3)$$

which specifies the angular speed of A around B . Since ω is the angular speed,

$$\omega = \frac{d\theta}{dt}. \quad (4)$$

Combining Eq. 3 and Eq. 4, we obtain the differential equation

$$\frac{d\theta}{dt} = 2v \sin \theta \quad (5)$$

which specifies how angle θ changes with time. With the translational motion with B taken into account, this differential equation specifies the trajectory of A before θ completes the change from α to β .

The differential equation can be solved by direct integration, such that

$$\int_{\alpha}^{\beta} \frac{1}{\sin \theta} d\theta = \int_0^t 2v dt. \quad (6)$$

Using the formula

$$\int \frac{1}{\sin \theta} d\theta = \ln[\tan(\theta/2)] \quad (7)$$

we have

$$2vt = \ln[\tan(\beta/2)] - \ln[\tan(\alpha/2)]. \quad (8)$$

Equation 8 gives the time t required for θ to change from α to β .

To complete the motion within time L/v ,

$$t \leq L/v \quad (9)$$

must hold. Thus from Eq. 8

$$\ln [\tan(\beta/2)] - \ln [\tan(\alpha/2)] \leq 2L. \quad (10)$$

Eq. 10 can also be written as

$$\tan(\beta/2) \leq \tan(\alpha/2)e^{2L}. \quad (11)$$

Eq. 11 is the basic result of this section. For AB to move to $A'B'$ in time L/v , where $|AB| = 1$ and $L = |BB'|$, angles α and β must satisfy the relation given by Eq. 11. The result implies that the difference in inclination of AB and $A'B'$ as represented by the ratio of $\tan(\beta/2)$ and $\tan(\alpha/2)$ cannot be overly large. Otherwise, robot A will not have enough time to complete its rotation. The time available to robot A to change angle θ from α to β is determined by L , the distance between B and B' .

Summarizing the discussion given above, we obtain the following theorem.

Theorem 1 *AB can be moved to $A'B'$ in optimal time L/v if and only if either*

1. $\alpha = \beta$, or
2. $0 < \alpha < \beta < 180^\circ$ and $\tan(\beta/2) \leq \tan(\alpha/2)e^{2L}$,

where L , α and β are as defined above.

2.2 Examples

We present examples to illustrate the method given in Subsection 2.1.

Given L , α and β , we first test if they satisfy Eq. 11. If Eq. 11 is satisfied, then we move robot B to B' along a straight line at speed v . The movement of A is a little complicated. When $\alpha \leq \theta < \beta$, we use Eq. 5 to find the angular speed of A around B , and calculate the net velocity of A by combining its rotational motion around B and the translational motion with B . When angle θ reaches β , A stops rotating around B , and moves towards A' along a straight line.

Figure 7 shows the movement obtained by this method for the case $L = 2$, $\alpha = 10^\circ$ and $\beta = 140^\circ$. The inequality ($\tan(\beta/2) < \tan(\alpha/2)e^{2L}$) in Eq. 11 holds in this case. Indeed, in the figure we observe that robot B moves along a straight line, and robot A moves along a curve first, and then along a straight line once angle θ reaches β (i.e., the ladder becomes parallel to $A'B'$). So robot A has more than enough time to complete the required angle change.

The next example, shown in Figure 8, is constructed by increasing the value of β to 156.35° while keeping parameters α and L from the previous example unchanged. This is a critical case of the applicability of the method, since the equality ($\tan(\beta/2) = \tan(\alpha/2)e^{2L}$) in Eq. 11 holds. This means that robot A has just enough time to complete the required rotation to arrive at the destination. Indeed, the figure shows that robots A and B reach their respective destinations simultaneously.

Finally, Figure 9 shows the case when β has been increased further to 170° . Eq. 11 no longer holds, and thus robot A has not reached its destination when B finishes its move towards B' .

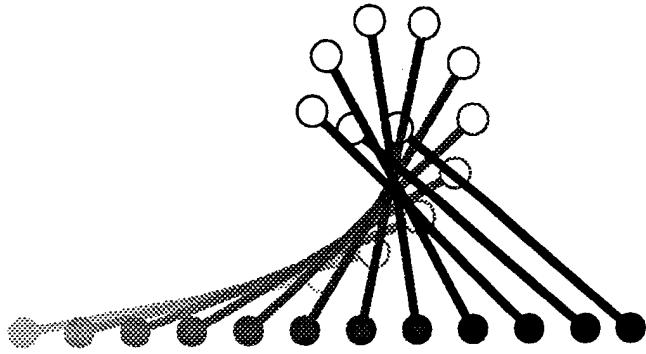


Figure 7: Segment AB moves to $A'B'$ in minimum time. The parameters are $L = 2$, $\alpha = 10^\circ$ and $\beta = 140^\circ$.

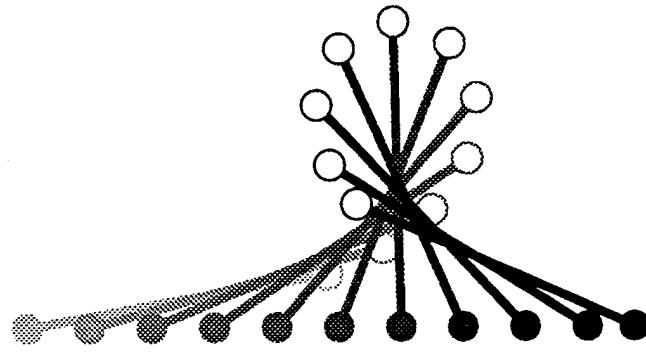


Figure 8: Optimal motion for the case $L = 2$, $\alpha = 10^\circ$ and $\beta = 156.35^\circ$. Compared to the previous example β is increased, so that robot A has just enough time to complete the rotation.

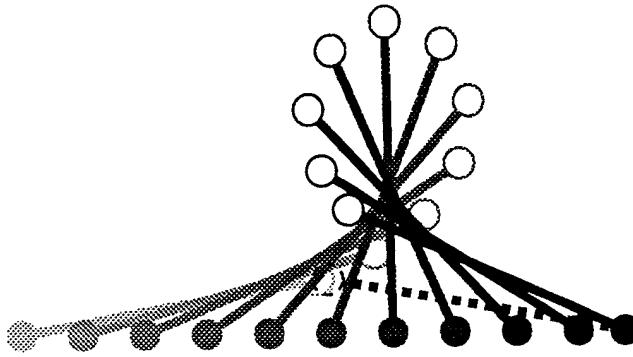


Figure 9: The case $L = 2$, $\alpha = 10^\circ$ and $\beta = 170^\circ$. Robot A does not have sufficient time to complete the rotation. The goal position $A'B'$ is shown in dashed line.

2.3 A Special Case

Consider a special case $\alpha + \beta = 180^\circ$, where $0 < \alpha < \beta < 180^\circ$. As is shown in Figure 10, let O be the intersection of the extensions of AB and $A'B'$. Let $\xi = |OB|$ and $\gamma = \angle BOB'$. Then AB and $A'B'$ are symmetrical about the line that bisects the angle γ .

Since in this case

$$\alpha = 90^\circ - \frac{1}{2}\gamma, \quad (12)$$

$$\beta = 90^\circ + \frac{1}{2}\gamma, \quad (13)$$

and

$$L = 2\xi \sin \gamma/2, \quad (14)$$

Eq. 11 can be rewritten as

$$\tan(45^\circ + \frac{\gamma}{4}) \leq \tan(45^\circ - \frac{\gamma}{4}) e^{4\xi \sin \gamma/2}. \quad (15)$$

Using the relations

$$\tan \frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}} \quad (16)$$

and

$$\cos(90^\circ \pm x) = \mp \sin x, \quad (17)$$

Eq. 15 can be simplified to

$$\frac{1 + \sin \gamma/2}{1 - \sin \gamma/2} \leq e^{4\xi \sin \gamma/2}. \quad (18)$$

Figure 11 is the curve of γ versus ξ for the equality ($\frac{1 + \sin \gamma/2}{1 - \sin \gamma/2} = e^{4\xi \sin \gamma/2}$) in Eq. 18. The region below the curve represents the combinations of ξ and γ for which the method in Subsection 2.1 is applicable and generates a motion that uses optimal time L/v , where $L = |BB'|$. In particular, the case $\xi = 0.5$ is a pure rotation around the mid-point of AB ,

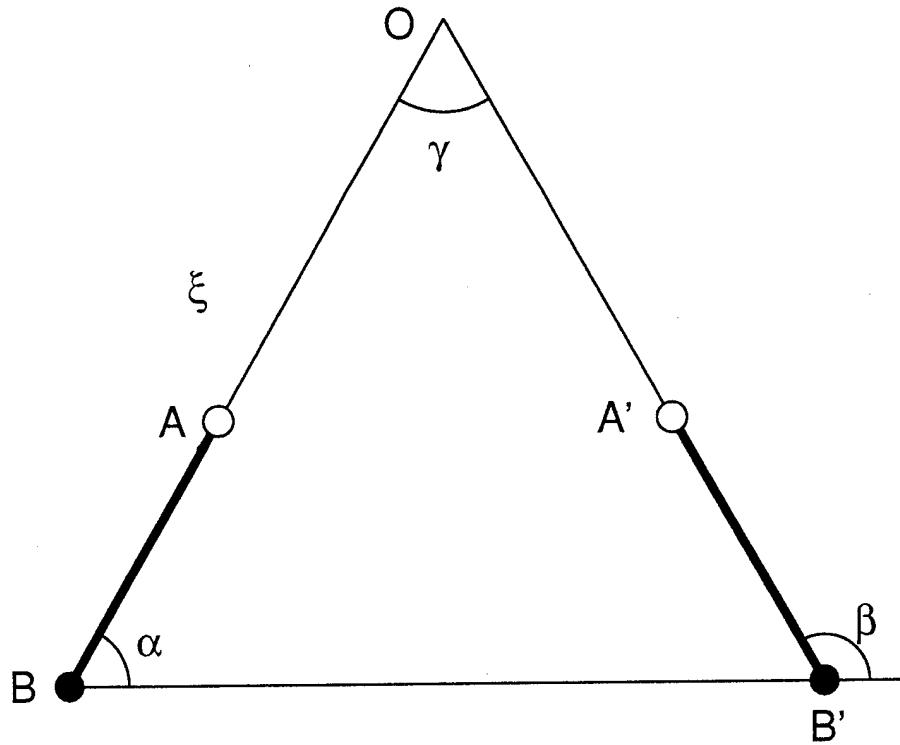


Figure 10: Illustration of the symmetric case. $\alpha + \beta = 180^\circ$.

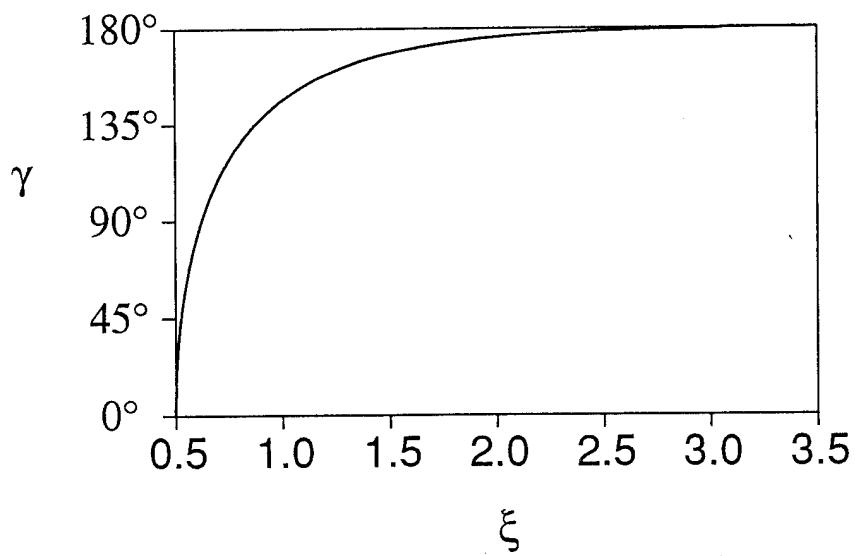


Figure 11: The curve of γ versus ξ , where $\xi = |OB|$ and $\gamma = \angle BOB'$.

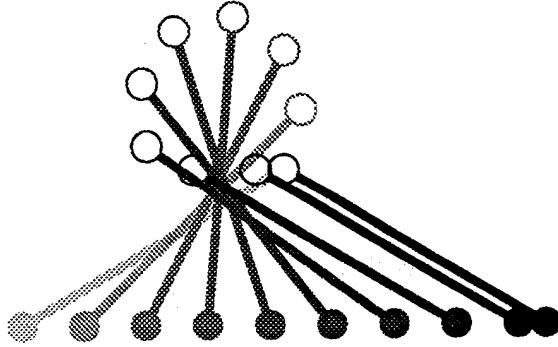


Figure 12: Optimal motion for AB to make an angle change around A by 120° , i.e., $\xi = 1$ and $\gamma = 146.48^\circ$. Robot A returns to its original position at the end of the motion.

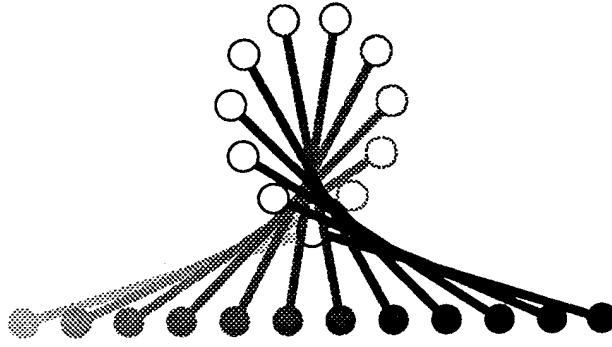


Figure 13: Optimal motion for $\xi = 1$ and $\gamma = 146.48^\circ$.

for which $\gamma = 0$, implying that the method is not applicable. The case $\xi = 1$ corresponds to $A = A'$, or a “rotation” of AB around A , and the method of this section (in which B moves straight to B' at speed v) can be used for the values of γ in the range between 0 and 146.48° . As ξ increases, the largest possible value of γ approaches 180° . By Theorem 1, this limit ($\gamma = 180^\circ$, hence $\alpha = 0$ and $\beta = 180^\circ$) can never be reached.

Figure 12 shows the optimal motion of AB for $\xi = 1$ and $\gamma = 120^\circ$. In this case, robot A completes the necessary rotation before B reaches the destination, and then moves to A' (which coincides with its initial position since $\xi = 1$) along a straight line.

In Figure 13 we increase γ to 146.48° while keeping $\xi = 1$. As we mentioned above, this is a critical case for $\xi = 1$, and thus A completes its rotation exactly when B reaches B' .

Figure 14 shows the movements of the robots using the same strategy for the case $\xi = 1$ and $\gamma = 160^\circ$ ($> 146.48^\circ$). In this case robot A does not have enough time to complete the rotation, and in fact, the last position of A shown in Figure 14 is not its final position.

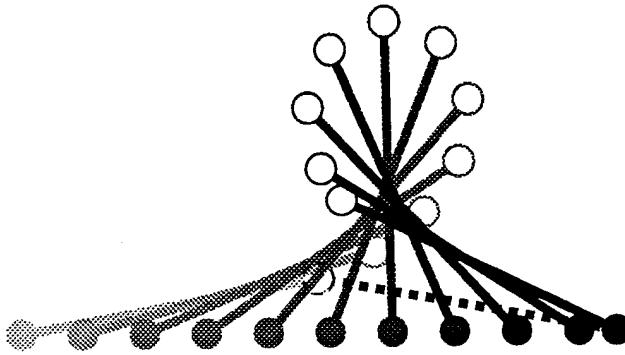


Figure 14: The case $\xi = 1$ and $\gamma = 160^\circ$. Robot A cannot complete the rotation. The goal position $A'B'$ is shown in dashed line.

2.4 A General Property of Optimal Motion

As we stated before, we assume that at any moment during the motion, the speed of each robot is either a given constant v or 0. So we allow a robot to remain stationary while the other robots rotates around it along an arc. We now prove that both robots must *always* move at speed v in any optimal motion.

Theorem 2 *In any optimal motion, both robots move at speed v at any moment.*

Proof Suppose in an optimal motion, robot A remains stationary over an interval of time, and thus robot B rotates around robot A along an arc. (Clearly, if neither robot moves over an interval, then the motion is not optimal.) See Figure 15. We can choose the interval to be sufficiently small, so that B rotates around A for an angle γ less than 146.48° . Then the motion in this interval is exactly the one discussed in Subsection 2.3, with $\xi = 1$ and $\gamma < 146.48^\circ$. So the optimal motion in this interval is the one in which B moves straight to its destination while A moves along a circular path. Clearly this motion takes less time than the original one. This is a contradiction. \square

3 The Variational Method

As we have seen in Section 2, if Eq. 11 is satisfied then the best strategy is to move the robot B further away from its destination along a straight line, and rotate robot A around it. On the other hand, to move optimally when Eq. 11 is not satisfied, intuitively robot B will have to move along a curve. This will yield some extra time for robot A to complete the necessary rotation around B .

In this section we consider the case when Eq. 11 is not satisfied, and discuss a method for selecting an optimal path for B (and hence an optimal motion for A and B). The method we use is known as variational calculus, a branch of functional analysis. In the next subsection, we give an outline of the method.

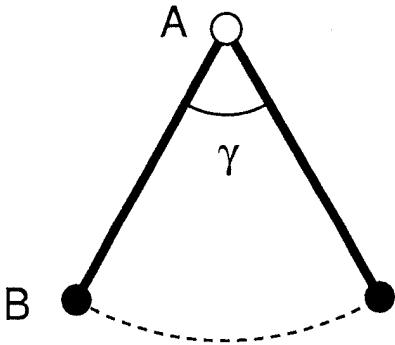


Figure 15: **Robot *B*** rotates around robot *A* that remains stationary during the motion. The motion is not optimal.

3.1 Variational Calculus

Different from ordinary functions, a functional is a quantity which depends on the entire course of one or more functions [2]. In other words, the domain of a functional is a set of functions rather than a region of a coordinate space. For example, in our study of the motion of the two robots, the path of robot *B* can be described by a function, and the length of the path is a functional which is determined by the form of the path.

Variational calculus deals with the extrema of a functional [5]. By applying variational calculus, one can find the minima or maxima of a functional, which in turn determines the form of the function that produces such extreme values. To find the extrema of a functional, **one first** identifies independent variables of the functional which are functions or derivatives of functions. A functional has the property that it is invariant under the first order change of its variables [5]. This property leads to a set of differential equations known as the Euler equations. By solving the Euler equations, the functions that maximize the functional can be found. Often, the solution is associated with specific boundary conditions of these functions.

A typical example illustrating an application of variational calculus is finding the shortest path between two points in a plane. As we know the solution is a straight line, which is indeed the solution to the Euler equation. For our present problem, the point designated by robot *B* cannot move arbitrarily. Rather, it must move in such a way that the distance from robot *A* to robot *B* remains constant. Subsidiary conditions like this are called constraints and can be handled systematically by using the notion of Lagrange multiplier [2, 5]. The method we use in this section that involves Lagrange multiplier and Euler equations is called the Euler-Lagrange formulation.

3.2 The Coordinate System

We set up a Cartesian coordinate system with reference to the initial and final positions of the robots, *A*, *B*, and *A'*, *B'* with $|BB'| \geq |AA'|$. The *x*-axis is chosen to be along BB' with origin at *B*, and the *y*-axis is perpendicular to BB' as is shown in Figure 16. Under this

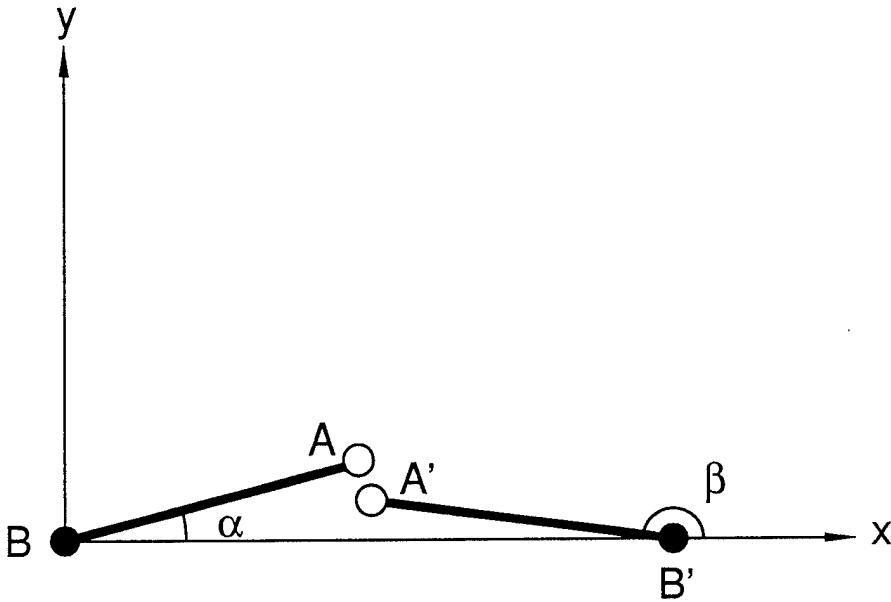


Figure 16: The setup of the Cartesian coordinate system with reference to the initial and final positions of the robots.

coordinate system, B' is at $(L, 0)$ with $L = |BB'|$, and α and β , respectively, are the angles that segments AB and $A'B'$ make with the x -axis. We denote by θ the angle between AB and the x -axis during the motion. First, we give a general property any optimal motion of the two robots.

Lemma 2 *In any optimal motion, θ changes monotonically.*

Proof If θ does not change monotonically, then there must be three positions of the segment (occurring in this order) such that the first and the third have an identical inclination different from that of the second. Then the motion can be shortened by replacing the portion between the first and the third by a pure translation (see Figure 17). \square

The property of optimal motion given in Lemma 2 applies also to the case considered in Section 2.

By Lemma 2, during any optional motion either $\dot{\theta} \geq 0$ (segment AB rotates counter-clockwise about B), or $\dot{\theta} \leq 0$ (segment AB rotates clockwise about B). The overall optimal motion is one of the respective optimal motions for the two cases. To find an optimal motion for the first case, we take α and β so that $\alpha < \beta < \alpha + 360^\circ$, and for the second case, we take $\alpha - 360^\circ < \beta < \alpha$. In the rest of this section we only discuss the case $\alpha < \beta < \alpha + 360^\circ$. The discussion for the other case is similar and is omitted.

Lemma 2 does not state that the change of θ be strictly monotonous. Indeed, it is possible that an optimal motion contains pure translation (see, for instance, Figure 12). However, that is no longer possible for the case considered in this section, as the next lemma states.

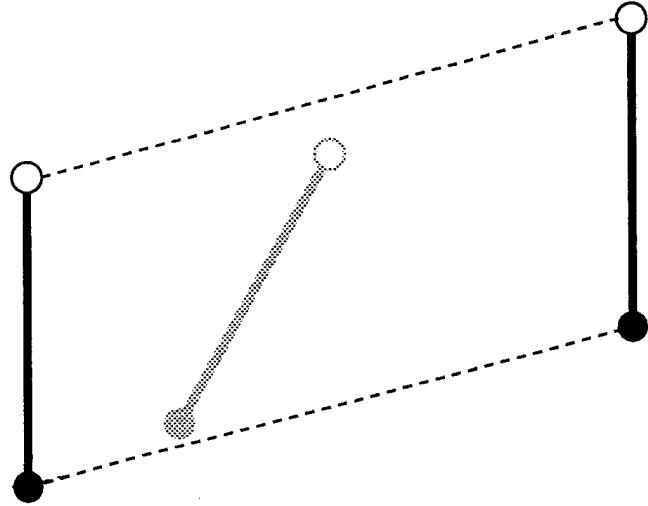


Figure 17: Illustration showing the situation considered in the proof of Lemma 2. The trajectories shown in dashed lines are optimal. The intermediate position of the segment with different inclination should be avoided.

Lemma 3 *If Eq. 11 is not satisfied, then $\dot{\theta}$ changes strictly monotonously in any optimal motion.*

Proof We only give an outline. By Lemma 2, suppose $\dot{\theta} \geq 0$ during the given optimal motion. (The case $\dot{\theta} \leq 0$ is similar and is omitted.) Since Eq. 11 is not satisfied, there is a section of the motion in which $\dot{\theta} > 0$, and the trajectory of B cannot be a single straight line. Suppose there are sections of the motion in which $\dot{\theta} = 0$. In these sections the robots simply travel along parallel straight lines (rather than parallel curves), since otherwise the motion cannot be optimal.

Consider the portion of the trajectory of B where a section, say s , with $\dot{\theta} > 0$ meets a section, say s' , with $\dot{\theta} = 0$. We may assume s precedes s' . (Otherwise we consider the reversal of the given motion. Clearly, the reversal of any optimal motion from AB to $A'B'$ is an optimal motion from $A'B'$ to AB .) Section s may be a curve, but s' is a straight line segment. See Figure 18. Assume either s' is a curve, or s and s' meet at an angle. Suppose we replace, by a straight line segment, an arbitrary small portion of ss' including the point where s and s' meet. Then travel time of B is reduced, but as a side effect, the increase of θ achieved in s may be reduced by an arbitrary small amount. This will produce an extra increase of θ that needs to be handled in section s' . However, since this extra increase can be made arbitrarily small, section s' can absorb it readily, reducing the overall time needed for the motion. This is a contradiction to the assumption that the given motion is optimal.

Suppose that s' is a straight line segment, and s and s' together form a longer straight line segment. Then since the trajectory of B cannot be a single line segment, the segment formed by merging adjacent sections must eventually meet a section with $\dot{\theta} > 0$ that is either a curve or that forms a corner with the segment. Then the argument given in the previous paragraph can be applied. This completes the proof. \square

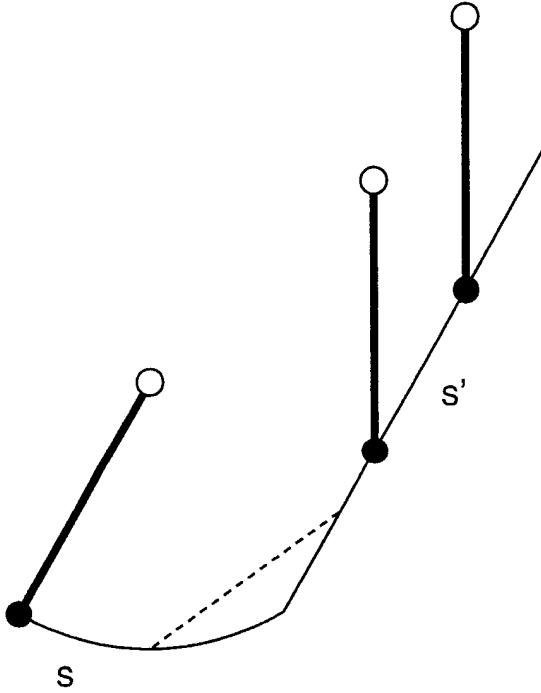


Figure 18: Illustration for the proof of Lemma 3. The dashed line denotes a shortcut.

3.3 Motion Constraints

To obtain the optimal path for the robots, we need to minimize the length of the trajectory of B (or A) from its initial position to its final position. If we denote by (x, y) the coordinate of B during the motion, then the length element of the trajectory of B can be written as

$$\sqrt{\dot{x}^2 + \dot{y}^2} dt \quad (19)$$

where the dot on top of x and y denotes differentiation with respect to time t . Integrating Eq. 19 over time gives the length of the trajectory of B which is the quantity to be minimized. The resulting integral is a functional with \dot{x} and \dot{y} as variables. In this section we will discuss the constraints that must be met by the optimization procedure.

There are two constraints for the motion of robots A and B . First, the distance between them must remain constant, $|AB| = 1$, during the motion. The second constraint is that by Theorem 2, both robots must move at constant speed v at any moment. We now find the consequences of these constraints. Since B moves along a curve, we denote by ϕ the angle between the x -axis and the direction of the velocity of B . See Figure 19. Recall that θ is the angle between the x -axis and BA . From the definition of ϕ

$$\frac{dy}{dx} = \tan \phi. \quad (20)$$

Relative to robot B , robot A rotates with angular speed ω , where

$$\omega = \frac{d\theta}{dt}. \quad (21)$$

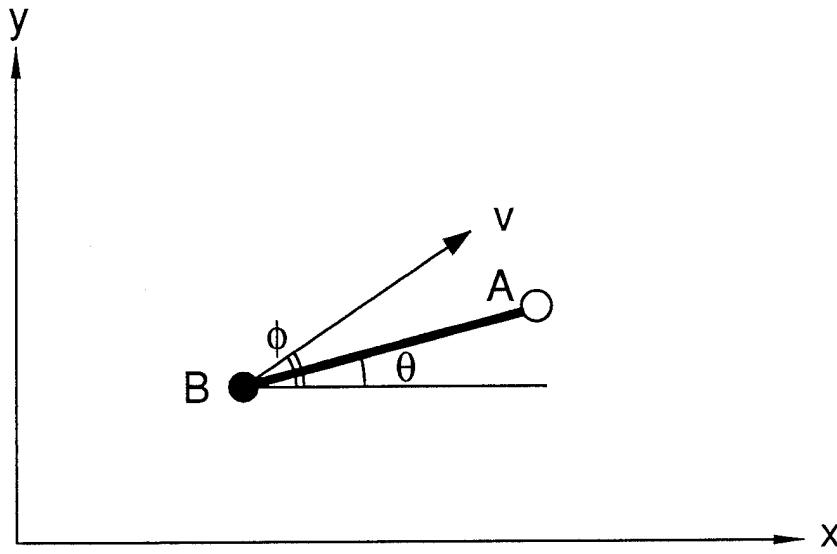


Figure 19: The definitions of θ and ϕ . The arrow represents the velocity of robot B .

The angle between the velocity of B and the velocity of A due to rotation around B is $90^\circ + \theta - \phi$. Therefore V , the resultant speed of A , is given by

$$V^2 = v^2 + \omega^2 - 2\omega v \sin(\theta - \phi). \quad (22)$$

Equation 22 is a consequence of the first constraint on the motion of the robots. For the second constraint, we have $V = v$. It then follows from Eq. 22 that for $\omega > 0$,

$$\begin{aligned} \omega &= 2v \sin(\theta - \phi) \\ &= 2v \sin \theta \cos \phi - 2v \cos \theta \sin \phi. \end{aligned} \quad (23)$$

It should be noted that the case with $\omega = 0$ also satisfies the motion constraint. However, by Lemma 3, the case with $\omega = 0$ will not be present in an optimal solution.

From the definition of ϕ , we represent the components of the velocity of B in terms of v and ϕ as

$$\frac{dx}{dt} = v \cos \phi \quad (24)$$

and

$$\frac{dy}{dt} = v \sin \phi. \quad (25)$$

Substituting Eq. 24 and Eq. 25 into Eq. 23, we have

$$\dot{\theta} - 2\dot{x} \sin \theta + 2\dot{y} \cos \theta = 0 \quad (26)$$

where the dot over x , y and θ represents derivative with respect to time. Equation 26 describes the constraints on the optimal motion.

Figure 20 gives an illustration of the constraints. As shown in the figure, under the constraints, the components of the velocities of the robots parallel to AB must be equal so that there is no relative motion along the line joining A and B . In addition, the components of the velocities of the robots perpendicular to AB are equal in magnitude and opposite in direction to each other. As a result, the direction of the center of mass motion of the robots is always aligned with the line segment AB .

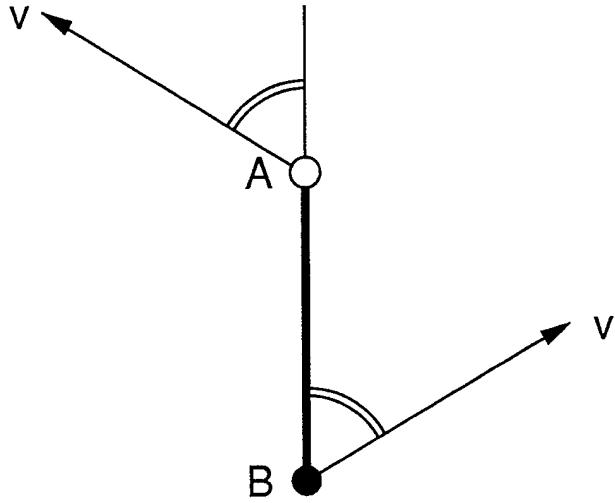


Figure 20: Illustration of the motion constraints. The arrows represent the velocities of the robots. The velocities have identical magnitude, and their angles from the segment BA are also identical.

3.4 Euler-Lagrange Formulation

We now apply the variational method to find the differential equations that the optimal path obeys.

According to the results in Subsection 3.3, our task is to minimize the integral

$$F = \int_0^t \sqrt{\dot{x}^2 + \dot{y}^2} dt \quad (27)$$

subject to the constraint

$$\dot{\theta} - 2\dot{x} \sin \theta + 2\dot{y} \cos \theta = 0 \quad (28)$$

and the boundary conditions

$$\begin{cases} x(0) = 0 \\ y(0) = 0 \end{cases} \quad (29)$$

$$\begin{cases} x(t) = L \\ y(t) = 0 \end{cases} \quad (30)$$

and

$$\theta(t) - \theta(0) = \beta - \alpha. \quad (31)$$

From the theory of functional variations, we know that minimizing the integral given by Eq. 27 with the constraint Eq. 28 is equivalent to minimizing the following expression [2]

$$\int_0^t L(\dot{x}, \dot{y}, \dot{\theta}, \theta) dt \quad (32)$$

where

$$L(\dot{x}, \dot{y}, \dot{\theta}, \theta) = \sqrt{\dot{x}^2 + \dot{y}^2} + \lambda(t)(\dot{\theta} - 2\dot{x} \sin \theta + 2\dot{y} \cos \theta). \quad (33)$$

In Eq. 33, $\lambda(t)$ is so called the Lagrange multiplier and it is a function of time [2]. Minimization of Eq. 32 is also required to satisfy the boundary conditions Eq. 29, Eq. 30 and Eq. 31.

The application of the Euler-Lagrange formulation yields the following set of equations

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0 \quad (34)$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = 0 \quad (35)$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = 0. \quad (36)$$

In addition, the optimal path must satisfy the constraint Eq. 28 and

$$\dot{x}^2 + \dot{y}^2 = v^2. \quad (37)$$

It should be noted that among Eq. 34, Eq. 35 and Eq. 36, only two equations are independent [8]. For convenience, we will use only Eq. 34 and Eq. 35, together with Eq. 28 and Eq. 37.

3.5 The Solution

Since L is not an explicit function of x and y , the first term on the left hand side of Eq. 34 and Eq. 35 is zero. The second term of the Eq. 34 and Eq. 35 gives

$$\frac{d}{dt} \left[\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} - 2\lambda(t) \sin \theta \right] = 0 \quad (38)$$

$$\frac{d}{dt} \left[\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} + 2\lambda(t) \cos \theta \right] = 0. \quad (39)$$

Integrating both sides of Eq. 38 and Eq. 39, we have

$$\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} - 2\lambda(t) \sin \theta = a \quad (40)$$

$$\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} + 2\lambda(t) \cos \theta = b \quad (41)$$

where a and b are constant. Applying Eq. 37 we have

$$\frac{\dot{x}}{v} - 2\lambda(t) \sin \theta = a \quad (42)$$

$$\frac{\dot{y}}{v} + 2\lambda(t) \cos \theta = b. \quad (43)$$

To eliminate the unknown function $\lambda(t)$, we multiply Eq. 42 and Eq. 43 by $\cos \theta$ and $\sin \theta$, respectively. Adding the results, we obtain

$$\dot{x} \cos \theta + \dot{y} \sin \theta = av \cos \theta + bv \sin \theta. \quad (44)$$

Solving for \dot{x} and \dot{y} from Eq. 44 and the constraint, Eq. 28, we have

$$\dot{x} = \frac{\dot{\theta}}{2} \sin \theta + av \cos^2 \theta + bv \sin \theta \cos \theta \quad (45)$$

$$\dot{y} = -\frac{\dot{\theta}}{2} \cos \theta + av \cos \theta \sin \theta + bv \sin^2 \theta. \quad (46)$$

Substituting these expressions for \dot{x} and \dot{y} into Eq. 37, and simplifying the result, we have

$$\frac{\dot{\theta}^2}{4} + (av \cos \theta + bv \sin \theta)^2 = v^2. \quad (47)$$

In Eq. 45, Eq. 46 and Eq. 47, a and b are parameters to be determined from the boundary conditions. We can replace a and b by two equivalent parameters c and δ defined by

$$c = \sqrt{a^2 + b^2} \quad (48)$$

and

$$\tan \delta = \frac{a}{b}. \quad (49)$$

With the newly defined constants, Eq. 45, Eq. 46 and Eq. 47 can be written as

$$\dot{x} = \frac{\dot{\theta}}{2} \sin \theta + cv \cos \theta \sin(\theta + \delta) \quad (50)$$

$$\dot{y} = -\frac{\dot{\theta}}{2} \cos \theta + cv \sin \theta \sin(\theta + \delta) \quad (51)$$

$$\dot{\theta} = 2v \sqrt{1 - c^2 \sin^2(\theta + \delta)}. \quad (52)$$

Eq. 50, Eq. 51 and Eq. 52 are the basic result of this section. From Eq. 50 and Eq. 51, we can find the instantaneous velocity of robot B , hence its trajectory. Equation 52 gives the angular speed of robot A around robot B .

To determine the parameters c and δ , we divide both sides of Eq. 50 and Eq. 51 by $\dot{\theta}$, using Eq. 52. The two equations then become

$$\frac{dx}{d\theta} = \frac{1}{2} \sin \theta + \frac{c \cos \theta \sin(\theta + \delta)}{2\sqrt{1 - c^2 \sin^2(\theta + \delta)}} \quad (53)$$

$$\frac{dy}{d\theta} = -\frac{1}{2} \cos \theta + \frac{c \sin \theta \sin(\theta + \delta)}{2\sqrt{1 - c^2 \sin^2(\theta + \delta)}}. \quad (54)$$

Integrating both sides of Eq. 53 and Eq. 54 from $\theta = \alpha$ to $\theta = \beta$, we have

$$\frac{1}{2} \int_{\alpha}^{\beta} \sin \theta d\theta + \frac{c}{2} \int_{\alpha}^{\beta} \frac{\cos \theta \sin(\theta + \delta)}{\sqrt{1 - c^2 \sin^2(\theta + \delta)}} d\theta = L \quad (55)$$

$$-\frac{1}{2} \int_{\alpha}^{\beta} \cos \theta d\theta + \frac{c}{2} \int_{\alpha}^{\beta} \frac{\sin \theta \sin(\theta + \delta)}{\sqrt{1 - c^2 \sin^2(\theta + \delta)}} d\theta = 0. \quad (56)$$

To obtain Eq. 55 and Eq. 56, we have used the boundary conditions given by Eq. 29, Eq. 30 and Eq. 31. The two equations are sufficient for determining the two constants c and δ . From these two constants and Eq. 50, Eq. 51 and Eq. 52 we can find the path of the optimal motion.

3.6 Results in Terms of Elliptic Integrals

It is possible to express our results in terms of the Legendre elliptic integrals [7] defined as

$$F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad (57)$$

$$E(\phi, k) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta} d\theta. \quad (58)$$

$F(\phi, k)$ and $E(\phi, k)$ are known as the Legendre elliptic integral of the first and second kind, respectively. Using these definitions, Eq. 55 and Eq. 56 can be written as

$$\begin{aligned} & \frac{1}{2}(\cos \alpha - \cos \beta) + \frac{\cos \delta}{2c} \left[\sqrt{1 - c^2 \sin^2(\alpha + \delta)} - \sqrt{1 - c^2 \sin^2(\beta + \delta)} \right] \\ & + \frac{\sin \delta}{2c} [F(\beta + \delta, c) - F(\alpha + \delta, c) - E(\beta + \delta, c) + E(\alpha + \delta, c)] = L \end{aligned} \quad (59)$$

and

$$\begin{aligned} & \frac{1}{2}(\sin \alpha - \sin \beta) - \frac{\sin \delta}{2c} \left[\sqrt{1 - c^2 \sin^2(\alpha + \delta)} - \sqrt{1 - c^2 \sin^2(\beta + \delta)} \right] \\ & + \frac{\cos \delta}{2c} [F(\beta + \delta, c) - F(\alpha + \delta, c) - E(\beta + \delta, c) + E(\alpha + \delta, c)] = 0. \end{aligned} \quad (60)$$

Once given α , β and L , these two equations can be solved numerically to yield c and δ . We can then use the values to produce the optimal path from Eq. 50, Eq. 51 and Eq. 52.

We can also calculate t , the time span of the optimal motion, using Eq. 52 as follows.

$$\begin{aligned} t &= \int_\alpha^\beta \frac{dt}{d\theta} d\theta \\ &= \frac{1}{2v} \int_\alpha^\beta \frac{d\theta}{\sqrt{1 - c^2 \sin^2(\theta + \delta)}}. \end{aligned} \quad (61)$$

In terms of elliptic integral we then have

$$t = \frac{1}{2v} [F(\beta + \delta, c) - F(\alpha + \delta, c)]. \quad (62)$$

The notation of elliptic integrals not only makes our expressions look more compact, it also allows us to apply available algorithms for calculating $F(\phi, k)$ and $E(\phi, k)$. The algorithms make use of the properties of the elliptic integrals, and converge very fast [12]. We used these algorithms to compute optimal motion for the examples given in the next section.

Summarizing the discussion of this section, we obtain:

Theorem 3 Suppose that Eq. 11 is not satisfied. An optimal motion in which segment AB rotates about B counterclockwise can be obtained from Eq. 50, Eq. 51 and Eq. 52, using c and δ satisfying Eq. 59 and Eq. 60. The time needed to execute the motion is obtained from Eq. 62.

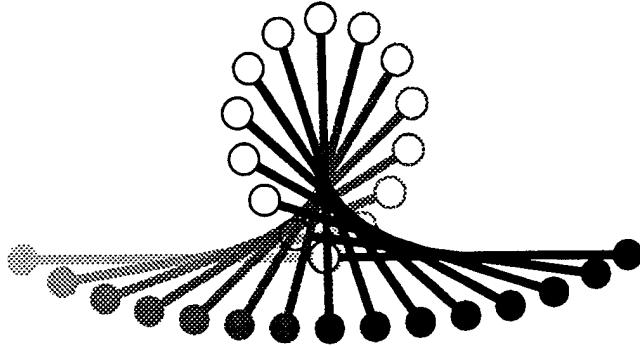


Figure 21: Optimal motion in which segment AB turns around by 180° . Parameters are $L = 2$, $\alpha = 0$ and $\beta = 180^\circ$.

3.7 Examples

Figure 21 shows an optimal motion obtained by the method presented in this section, for the case when segment AB must be turned around by 180° about A . The parameters for this instance are $L = 2$, $\alpha = 0$ and $\beta = 180^\circ$, and the method in Subsection 2.1 is not applicable. This motion has been obtained as follows.

We first substitute the value of L , α and β into Eq. 59 and Eq. 60. Solving the two equations for c and δ numerically, we obtain $\delta = 90^\circ$ and $c = 0.836$. Using these values and Eq. 50, Eq. 51 and Eq. 52, we generate the trajectories of the two robots, shown in Figure 21. From Eq. 62 the length of the trajectory of each robot is found to be 2.074.

Figure 22 shows a suboptimal motion to the same instance obtained by applying the method of Section 2 twice. The motion consists of two phases. In the first phase, the angle θ changes by 90° while B moves along a straight line to an intermediate point C . At this moment the segment is vertical. Point C is chosen so that it is symmetrical with respect to B and B' , and is as close to the segment BB' as possible while keeping the method of Section 2 applicable. That is, if we let $L' = |BC|$, $\alpha' = \angle ABC$ and $\beta' = 180^\circ - \angle ACB$, then we have, from Eq. 11,

$$\tan(\beta'/2) = \tan(\alpha'/2)e^{2L'}. \quad (63)$$

With the additional relations

$$\beta' = 90^\circ + \alpha' \quad (64)$$

and

$$L' = \frac{1}{\cos \beta'} \quad (65)$$

we obtain $\angle ABC = 19.48^\circ$, which specifies the location of point C . The second phase is symmetric to the first case. That is, robot B that is currently at point C moves along a straight line to B' . The length of the trajectory of each robot in this suboptimal motion is 2.122, slightly larger than that of the optimal motion given above.

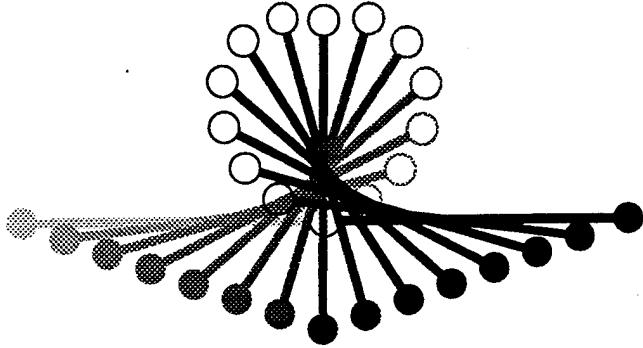


Figure 22: Sub-optimal motion for the instance of Figure 21 in which the segment turns around by 180° .

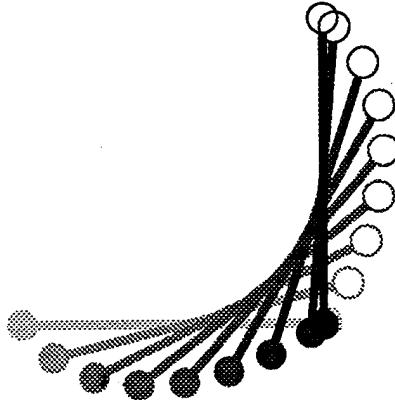


Figure 23: Optimal motion for $L = 1, \alpha = 0^\circ$ and $\beta = 90^\circ$.

The next example is for $L = 1, \alpha = 0^\circ$ and $\beta = 90^\circ$, which involves a rotation of 90° and translation. Figure 23 shows the result of our calculation. The length of the trajectory of each robot is approximately 1.1. It is easy to see that the solution to the Ulam problem mentioned in Section 1 (to minimize the sum of the lengths of total the trajectories of A and B) for this instance is to move A and B straight towards A' and B' , respectively. In our problem, however, B cannot move straight to its destination at speed v , since as we discussed in the proof of Lemma 1, such a move does not allow robot A to rotate.

Compare this to the instance shown in Figure 24, which involves only a rotation of 90° around A . The optimal motion shown in Figure 24, where the length of the trajectory of each robot is $\sqrt{2}$, has been obtained by the method of Section 2, since Eq. 11 is satisfied.

We conclude with two more examples. Figure 25 shows an optimal motion for the instance of Figure 9, where $L = 2, \alpha = 10^\circ$ and $\beta = 170^\circ$. Recall that the method

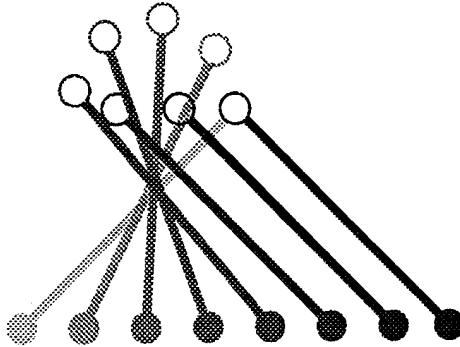


Figure 24: Optimal motion for $L = \sqrt{2}$, $\alpha = 45^\circ$ and $\beta = 135^\circ$ computed by the method of Section 2.

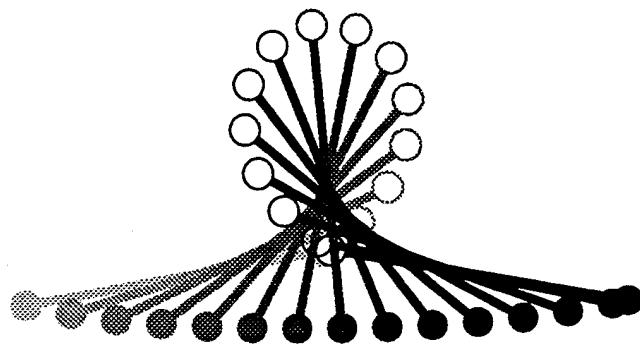


Figure 25: Optimal motion for the instance of Figure 9, where $L = 2$, $\alpha = 10^\circ$ and $\beta = 170^\circ$.

of Section 2 did not generate an optimal motion. The last example is the case $L = 2$, $\alpha = -10^\circ$ and $\beta = 120^\circ$. Figure 26 shows the result of our calculation.

4 Conclusion

We considered the problem of computing a time-optimal motion of two robots carrying a ladder, subject to the constraint that at any moment during the motion, the speed of each robot is either a given constant v , or 0. We identified an inequality, Eq. 11, that allows us to determine whether there exists an optimal motion in which the required displacement of the ladder is completed while the robot further away from its destination moves straight to its destination at speed v . If the inequality is not satisfied, then an optimal motion is computed using variational calculus.

In our discussions, a robot is not allowed to move at a nonzero speed less than v . Suppose

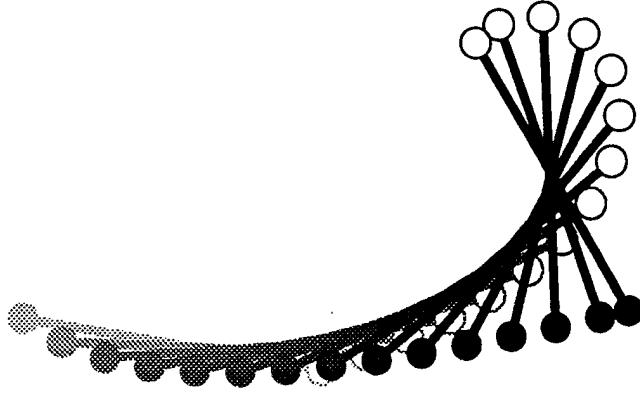


Figure 26: Optimal motion for $L = 2$, $\alpha = -10^\circ$ and $\beta = 120^\circ$.

we relax this condition and assume that each robot can move at any speed not exceeding v . If Eq. 11 is satisfied, then the motion computed in Section 2 is still optimal under this new assumption, since there is no “need” for the robots to slow down to achieve an optimal motion. If Eq. 11 is not satisfied, however, the results given in Section 3 based on variational calculus cannot be applied directly. To understand the difficulty in the analysis, consider robots A and B moving at instantaneous speeds v' and v'' , respectively, during an optimal motion, where $0 \leq v, v' \leq v$. Here, $\max\{v', v''\} = v$ holds, since otherwise we can reduce the time by increasing v' and v'' proportionally until one reaches v . Imagine that we draw, on the plane, exactly those parts of the trajectory of A where A moves at speed v , plus those parts of the trajectory of B where B moves at speed v but A 's speed at that moment is less than v . Let S be the collection of the curves drawn as above. Then since v is a constant, the total time needed to execute the motion is proportional to the sum of the lengths of the curves in S . So, finding an optimal motion is equivalent to finding such S having the smallest total sum. However, since either of A and B can be the faster robot at any moment, S is only piecewise continuous. This makes the analysis using variational calculus difficult.

For future research, we suggest to extend our work to the case in which (1) the robot can move at any speed not exceeding a given constant, (2) other kinematic constraints, such as maximum acceleration, are considered, and (3) the plane is occupied with obstacles.

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